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COUPLED DYNAMICS OF THE SOLAR SYSTEM

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Summary

Many simple physical models could be used to demonstrate chaotic behavior in physical systems. Numerical models give more control over the experimental conditions and allow one to study systems that are closer to ideal conservative systems. Our understanding of the Solar system has been revolutionized by the finding that multiple planet systems are subject to chaotic dynamical processes. In extreme cases chaos can disrupt some orbital configurations. The Solar system provides a plethora of examples of chaotic motion. In order to chaos to occur there must be at least two interacting oscillators. In the solar system that interference is supplied by a third body. This interference manifests itself as zones of chaotic motion.

Dynamical system can behave in a very way even if the governing equations are very simple. Loosely speaking, an irregular motion of a deterministic system is termed “chaotic behavior”. Chaotic behavior can be characterized by the exponential divergence of initially nearby trajectories, or, equivalently, chaotic behavior is characterized by sensitive dependence on initial conditions. Chaotic behavior occurs in both conservative and dissipative systems. Dynamical systems often show a transition to large-scale chaotic behavior, as a parameter is varied the fraction of orbits that are chaotic can suddenly change.

All of these features can be illustrated with a physical pendulum: chaotic behavior, sensitive dependence on initial conditions, regular behavior, transition from mostly regular behavior to large-scale chaotic behavior and so on.

Many simple physical models could be used to demonstrate chaotic behavior in physical systems. Numerical models give more control over the experimental conditions and allow one to study systems that are closer to ideal conservative systems. Many numerical models are interesting: the double pendulum, the driven pendulum, spin-orbit coupling, dynamically coupled oscillators, etc.

The Solar system is very nearly a conservative system. We should expect that the Solar system and many of its subsystems do exhibit large-scale chaotic behavior. The motion of the planets is chaotic with a Lyapunov time of order 4 million years, the evolution of the orbit of Mercury is especially irregular. The obliquity of Mars varies between about 10 and 50 on a multimillion year timescale, in a highly irregular manner. The chaotic evolution of the obliquity of Mars drastically affect the climate of Mars.

Other dynamical phenomena in the solar system involve chaotic behavior such as the transport of short-period comets from the Kuiper belt, and the delivery of meteoritic material from the asteroid belt.

It is convenient to choose the pendulum as a basic oscillating system for initial investigations. The pendulum is a good model to a wide variety of physical phenomena such as charge-density wave transport, quantum-mechanical Josephson systems, Solar system dynamics, etc. At present, it is also a widely used basic paradigm for analysis of complex, irregular and chaotic oscillations. Biased, we could say that any phenomenon that can be observed in the pendulum is of considerable generality. We believe that the general problem of excitation of oscillations in different systems under an inhomogeneous action of a nonlinear force should be most benefically analyzed on the example of the pendulum.

Chaotic systems have many fascinating properties, and there is a good deal of evidence that much of nature is chaotic; the Solar system, for instance. It raises a lot of neat and nasty problems about how to understand dynamics from observations, and about what it means to make a good mathematical model of something. But it's not the whole of dynamics, and in some ways not even the most interesting part.

The role of the resonances may be illustrated by discussing the dynamics of a simple model system - the mathematical pendulum subjected to a constant gravitational field. Depending on its energy, there exist two qualitatively different kinds of the motion: rotations, for which the pendulum rotates in one direction, so the sign of the angular momentum is constant, or librations around a stable fix point, for which the sign of the angular momentum changes twice during every period of oscillations. Looking at the phase space diagram of the motion, in which any point in the plane (angle θ) generates a certain trajectory, (Fig. 1), there exist a curve which separates different kinds of the dynamics. It is called separatrix and it consists of two branches connected at the unstable fixed point, ($\theta = \pi$, $p = 0$). The stable fixed point of the system is located at ($\theta = 0$, $p = 0$), in the center of the area bounded by the separatrix. The pendulum pointing down, ($\theta = 0$), is stable and perturbed in any direction returns back to its initial position. On the other hand, the unstable fixed point corresponds to the pendulum situated 'upside down', so an arbitrarily small perturbation would drive it out of this position.

The separatrix with the unstable fixed point is crucial for emergence of chaos. If there exist (at least) two interacting oscillators in the system, a region of chaotic motion emerges in the system. Such an effect may be easily observed in a simplified model of a periodically kicked rotator, defined by the following Hamiltonian

$$H = \frac{p^2}{2} + K\sum_n \delta(t-n)\cos(\pi\theta) \qquad (1)$$

The system rotates without friction and gravitational field but is subjected to infinitely short periodic perturbations (kicks), which take place at t = 1, 2, 3 ..., in units of the kicking period. The dynamics depends on the kicking strength K: for $K = 0$ there is no perturbation and the system performs free rotations, corresponding to horizontal lines in the phase space (θ, p). For positive values of the parameter K the dynamics becomes more complex: resonances with frequencies commensurate with the kicking period emerge around stable periodic orbits – see Fig.2. There exist also unstable periodic orbits. In contrast with the case of the pendulum shown in Fig. 1, the infinitely thin separatrices are transformed into layers of chaotic motion of a finite volume, which grows with the parameter *K*. Chaotic layers are formed around every resonance, so they do occur at the resonant frequencies. If the kicking strength exceeds a critical value $K_c \approx 0.97$ the last horizontal curve (the so-called KAM torus) breaks down, all chaotic layers become connected, so the trajectories may diffuse in the phase space acquiring arbitrarily high momentum, (the system is kicked and the energy in the system is not conserved).

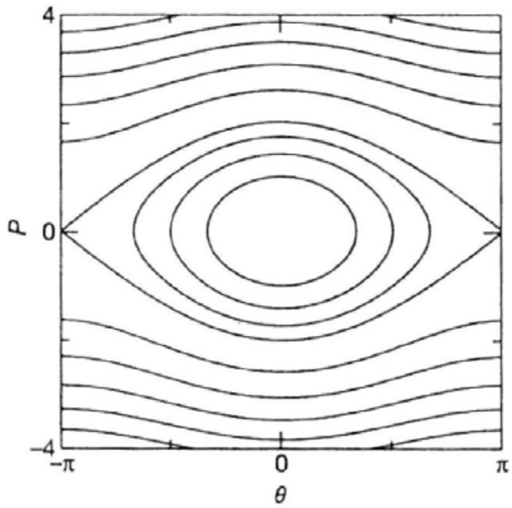

Figure 1 Phase space diagram for the pendulum resembles a cylinder. The elliptical fixed point (stable) in the center of the plot is surrounded by ellipses representing librations, while the hyperbolic fixed point (unstable) is located at the cut, ((θ = π,- π), and belongs to the separatrix. Curves encircling horizontaly the cylinder represent rotations.

Although there are no forces comparable to periodical kicking in the Solar System, as in the above toy model, the interference of the interactions of any body with the third body plays the very same role and is responsible for emergence of chaos. Furthermore, the destabilizing role of dynamical resonances may be observed by studying different issues of the Solar System. The famous Kirkwood gaps in the histogram of the density of asteroids plotted as a function of their semi-major axis may be explained by the interaction with Jupiter. This fact is apparent if the same data are used to produce a histogram as a function of the oscillation period: it shows minima at certain frequencies commensurate with the frequency of Jupiter. For instance, due to the 3:1 resonant interaction with Jupiter the trajectories of asteroids become unstable, which explains the observed minimum of the asteroids density at the frequency three times larger than the frequency of the motion of Jupiter.

The resonant interaction with Neptune influences the dynamics of bodies in the Kuiper Belt – the group of objects more distant from the Sun than Neptune, including Pluto. On the other hand, the resonant interaction between Saturn and Jupiter could contribute to the destabilization of the Solar system. In fact the actual frequencies of the both largest planets are close to the 5:2 resonance (Fig.3.). As shown by Michtchenko and Ferraz-Mello, a relatively small variation of the parameters determining the orbit of Jupiter would increase the role of the resonance, and in consequence would lead to the chaotic dynamics of both giant planets (Fig. 4).

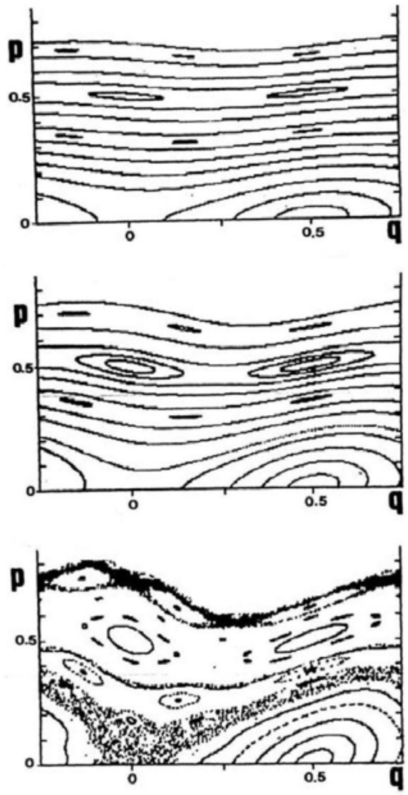

Figure 2 Phase space diagram for the periodically kicked rotator for kicking strength *a*) K = 0.25; *b*) K=0.51; *c*) $K = 1.02 > K_c$. Note an increase of the volume of the chaotic layer in the phase space.

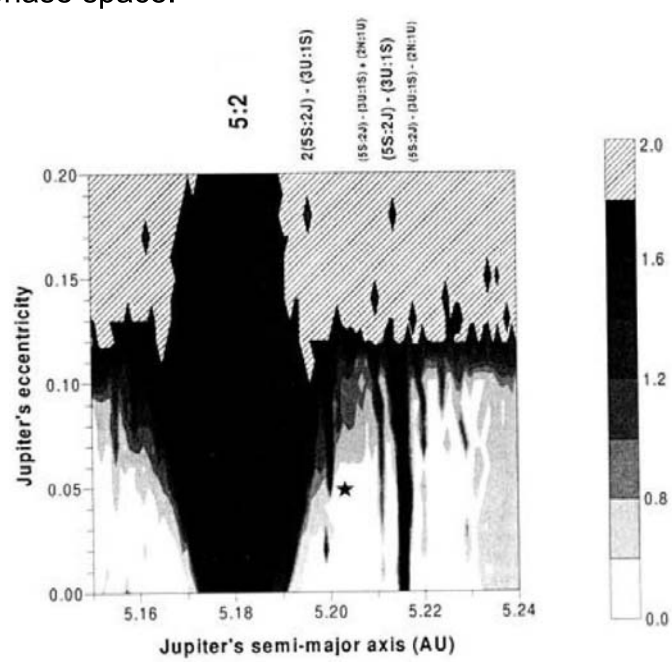

Figure 3 Dynamical map of the region around Jupiter: stability of the orbit initiated from a given point in the semi-major axis – eccentricity plane plotted in the grey scale – light (dark) region denotes regular (chaotic) motion while hatched region indicates orbits for which planetary collision occur. Star represents the actual position of the Jupiter, close to the 5:2 resonance.

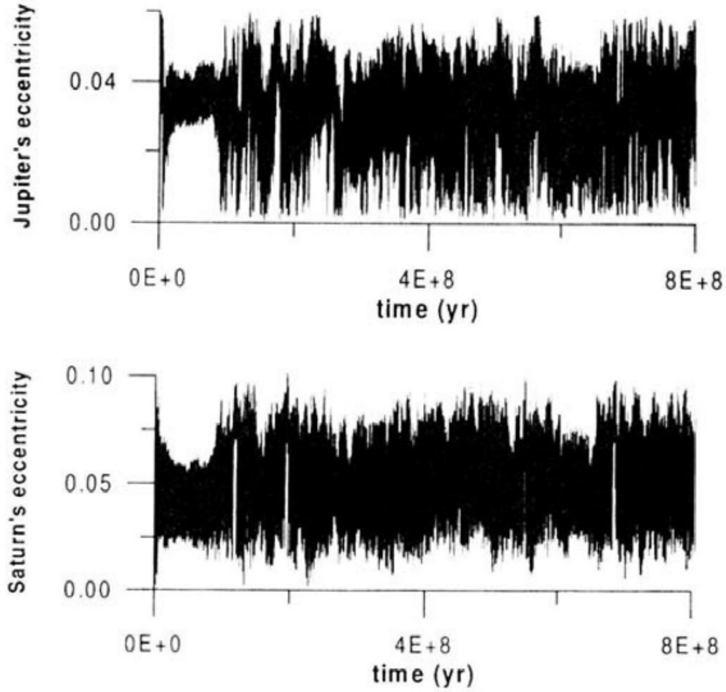

Figure 4 Chaotic evolution of Jupiter's and Saturn's eccentricity obtained by Michtchenko and S. Ferraz-Mello for modified initial conditions corresponding to 5:2 resonance.

Planets in the Solar system follow nearly Keplerian orbits. The orbit of each planet can be thought of as consisting of three nonlinear oscillators, corresponding to the three spatial directions. The Kepler problem is unusual in that all three oscillations have the same frequency. The orbital elements are chosen to take advantage of this degeneracy. The angle *l* varies on the orbital time scale, whereas the angle ω describing radial motion and the angle Ω describing vertical motion, are fixed. In the actual Solar system ω and Ω are time-dependent, with frequencies denoted by g_j and s_j, respectively. These frequencies are proportional to the mass ratios μ, and are consequently much smaller than the mean motion $n = dl/dt$, the time rate of change of the mean anomaly. Here *j* = 5, 6, 7 and 8 correspond to the radial order of the planets in the Solar system. The mean motions *n* (in units of cycles per day) and the modal frequencies of the Jovian planets can by determined by numerical integration of the equations of motion. Each planet's elements vary with all the frequencies *s* and *g*. In the case of Jupiter,

$$e_J \sin\omega_J \approx e_{55}\sin(g_5 t + \gamma_5) + e_{56}\sin(g_6 t + \gamma_6) + \dots, \quad (2)$$

where $e_{55} \approx 0.044$, $e_{56} \approx 0.016$ e 0.044, e 0.016, and $\gamma_{5,6}$ - constants.

A resonance occurs when two or more oscillators are coupled in such a way that a linear combination of their angles $\sigma = \sum_i p_i\theta_i$ *undergoes a bounded oscillation*, in which case σ is said to librate. In the sum defining σ, *i* denotes the *i*-th oscillator and the p_i's are (possibly negative) integers. When the oscillators are not resonant, all possible combinations of θ_i's increase or decrease in definitely, in which case is said to rotate.

The physical significance of a resonance is that energy is exchanged between the oscillators over a libration period, which is large compared to the oscillation period of any of the oscillators. This prolonged exchange can lead to large changes in the motion of the system. The orbit that divides regions of phase space where σ librates from those where σ rotates is called the separatrix.

The chaos in integrations of the outer planets arises from the overlap of the components of a three-body mean motion resonance among Jupiter, Saturn, and Uranus, with a minor role played by a similar resonance among Saturn, Uranus, and Neptune. The theory can by tested by using of numerical integrations. The width $\Delta a/a$ of the individual resonances are of order 3×10^{-6}, so that small changes in the initial conditions of the planets can lead to regular motion. However, the uncertainties in the initial conditions are smaller than the width of the individual resonances, so our Solar system is almost certainly chaotic. The resonance is extremely weak and hence easily disrupted.

Chaos in Hamiltonian systems, of which the motions of the planets are an example, arises when the separatrix of one resonance is perturbed by another resonance. The extent of the chaos depends on the parameter of stochasticity *K*, which is a function of the separatrix width divided by the distance between resonances. If *K* is small, there is little chaos, but for *K* > 1 the region in the immediate vicinity of the resonances is primarily chaotic. An orbit that at different times both librates and rotates, must cross the separatrix, and is therefore chaotic. Another signature of chaos is that two initially nearby chaotic orbits diverge exponentially with time.

Two planets are said to be in a mean motion resonance, when $p_1 \frac{d\lambda_1}{dt} \approx p_2 \frac{d\lambda_2}{dt}$.
In that case conjunctions between the planets occur at nearly fixed locations in space. The designation “mean motion” is a little misleading, because if $p_1 \neq p_2$ there is no coupling between the motion (λ, *a*) of two planets that does not involve a third degree of freedom, either the radial (ω, *e*) or vertical (Ω, *i*) motion of at least one of the planets.

There are no two-body mean motion resonances among the planets. However, there is a near–mean motion resonance between Jupiter and Saturn; Jupiter makes five circuits around the sun in about the same time that Saturn orbits twice. Saturn affects the orbit of Jupiter through its gravity, described by the potential

$$\phi = -\left(\frac{GM_s}{|\mathbf{r}_J - \mathbf{r}_S|}\right), \qquad (3)$$

where M_s is the mass of Saturn, $\mathbf{r}_J$ and $\mathbf{r}_S$ are the position vectors of Jupiter and Saturn, and *G* is the gravitational constant. To see the resonance mathematically, we expand $\mathbf{r}_J$ and $\mathbf{r}_S$ in terms of the orbital elements of the two planets up only to the lowest order terms:

$$\phi = -\left(\frac{GM_s}{a_s}\right)\sum\nolimits_{k,q,p,r} \phi^{(2,5)}_{k,q,p,r} \frac{a_S}{a_J} \times e_S^k e_J^q i_S^p i_J^r \cos(2\lambda_J - 5\lambda_S + $$
$$+ k\omega_s + q\omega_J + p\Omega_S + r\Omega_J) \qquad (4)$$

The amplitudes $\phi^{(2,5)}_{k,q,p,r}$ can be easily found in references. Simple considerations show that the integers in the argument of the cosine must sum to zero, or

$$2-5+k+q+p+r=0 \qquad (5)$$

and that $p+r$ must be even. This result shows *that the gravitational coupling between two bodies on Keplerian orbits always involves either* (ω, e) *or* (Ω, i)*, so that at least three oscillators are affected*. To lowest order in the eccentricities and inclinations, the integers *k*, *q*, *p* and *r* are non-negative and must sum to 3. The strength of the coupling is proportional to e^3 or ei^2, so this resonance is said to be of third order. Hence there are 10 frequencies associated with the resonance, four involving only perihelion precession rates, such as

$$2\dot{\lambda}_J - 5\dot{\lambda}_S + 2\dot{\omega}_J + \dot{\omega}_S \qquad (6)$$

and six involving the precession rates of the nodal lines, including

$$2\dot{\lambda}_J - 5\dot{\lambda}_S + \dot{\omega}_J + \dot{\Omega}_J + \dot{\Omega}_S \qquad (7)$$

The dot over the angles in these expressions denotes a time derivative. Each of the 10 members of Eq. 4 is referred to as a resonant term. The reason for this misuse of terminology is that, although none of the frequencies associated with these terms in our Solar system vanish, they are much smaller than the mean motions of Jupiter and Saturn. As a result, the resonant terms have a strong effect on the orbits of the two planets.

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